Solutions to graded problems in Homework 11

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9.5.15

Eigenvalues: The characteristic polynomial is:

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 0 \\ 2 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = (1-\lambda)((1-\lambda)(2-\lambda)-6)$$

$$= (1-\lambda)(\lambda^2-3\lambda-4) = (1-\lambda)(\lambda-4)(\lambda+1) = 0$$

which gives $\boxed{\lambda = -1, 1, 4}$.

Eigenvectors

$\lambda = -1$:

$$Nul(A+I) = Nul\begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix} = Nul\begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix} = Nul\begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = Span\left\{\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}\right\}$$

$\lambda = 1$:

$$Nul(A-I) = Nul\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix} = Nul\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} = Nul\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$= Nul\begin{bmatrix} 2 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} = Span\left\{\begin{bmatrix} \frac{1}{4} \\ \frac{-3}{2} \\ 1 \end{bmatrix}\right\} = Span\left\{\begin{bmatrix} 1 \\ -6 \\ 4 \end{bmatrix}\right\}$$

$\underline{\lambda = 4}$:

$$
\begin{aligned}
Nul(A-4I) = \quad & Nul \begin{bmatrix} -3 & 2 & 3 \\ 0 & -3 & 0 \\ 2 & 1 & -2 \end{bmatrix} = Nul \begin{bmatrix} -3 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix} = Nul \begin{bmatrix} -3 & 0 & 3 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix} \\
= \quad & Nul \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = Nul \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = Span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}
\end{aligned}
$$

<u>General Solution</u>

$$
\mathbf{x}(t) = Ae^{-t} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + Be^{t} \begin{bmatrix} 1 \\ -6 \\ 4 \end{bmatrix} + Ce^{4t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
$$

9.7.15

As usual $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_p$, where $\mathbf{x}_0$ is the general solution to the homogeneous equation $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x}_p$ is a particular solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

<u>Homogeneous solution:</u>

The characteristic polynomial is:

$$
(\lambda+4)(\lambda+1) - 4 = \lambda^2 + 5\lambda + 4 - 4 = \lambda^2 + 5\lambda = \lambda(\lambda+5) = 0
$$

which gives $\lambda = 0, -5$.

Moreover:

$$
Nul(A-0I) = Nul \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} = Nul \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = Span \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}
$$

and

$$
Nul(A+5I) = Nul \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = Nul \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = Span \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}
$$

Hence, the general solution to $\mathbf{x}' = A\mathbf{x}$ is:

$$
\begin{aligned}
\mathbf{x}_0(t) &= Ae^{0t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + Be^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad = A \begin{bmatrix} 1 \\ 2 \end{bmatrix} + Be^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\
&= A \begin{bmatrix} 1 \\ 2 \end{bmatrix} + B \begin{bmatrix} -2e^{-5t} \\ e^{-5t} \end{bmatrix}
\end{aligned}
$$

Particular solution

Suppose:

$$\mathbf{x}_p(t) = v_1(t)\begin{bmatrix}1\\2\end{bmatrix} + v_2(t)\begin{bmatrix}-2e^{-5t}\\e^{-5t}\end{bmatrix}$$

for functions v_1 and v_2 to be determined.

Then we have:

$$\mathbf{X}(t)\begin{bmatrix}v_1'(t)\\v_2'(t)\end{bmatrix} = \mathbf{f}(t)$$

Where:

$$\mathbf{X}(t) = \begin{bmatrix}1 & -2e^{-5t}\\2 & e^{-5t}\end{bmatrix}$$

is the fundamental matrix for the system (basically put the two vectors you found in $\mathbf{x}_p$ together in a matrix).

Then:

$$\begin{aligned}
\begin{bmatrix}v_1'(t)\\v_2'(t)\end{bmatrix} &= (\mathbf{X}(t))^{-1}\,\mathbf{f}(t)\\
&= \begin{bmatrix}1 & -2e^{-5t}\\2 & e^{-5t}\end{bmatrix}^{-1}\begin{bmatrix}\frac{1}{t}\\4+\frac{2}{t}\end{bmatrix}\\
&= \frac{e^{5t}}{5}\begin{bmatrix}e^{-5t} & 2e^{-5t}\\-2 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{t}\\4+\frac{2}{t}\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{5} & \frac{2}{5}\\\frac{-2}{5}e^{5t} & \frac{e^{5t}}{5}\end{bmatrix}\begin{bmatrix}\frac{1}{t}\\4+\frac{2}{t}\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{5t}+\frac{8}{5}+\frac{4}{5t}\\\cancel{\frac{-2e^{5t}}{5t}}+\frac{4}{5}e^{5t}+\cancel{\frac{2e^{5t}}{5t}}\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{t}+\frac{8}{5}\\\frac{4}{5}e^{5t}\end{bmatrix}
\end{aligned}$$

Therefore $v_1'(t) = \frac{1}{t} + \frac{8}{5}$, so:

$$v_1(t) = \int \frac{1}{t} + \frac{8}{5}dt = \ln|t| + \frac{8t}{5}$$

And $v_2'(t) = \frac{4}{5}e^{5t}$, so:

$$v_2(t) = \int \frac{4}{5}e^{5t}dt = \frac{4}{25}e^{5t}$$

Finally, we get:

$$\mathbf{x}_p(t) = v_1(t)\begin{bmatrix}1\\2\end{bmatrix} + v_2(t)\begin{bmatrix}-2e^{-5t}\\e^{-5t}\end{bmatrix} = \left(\ln|t| + \tfrac{8t}{5}\right)\begin{bmatrix}1\\2\end{bmatrix} + \left(\tfrac{4}{25}e^{5t}\right)\begin{bmatrix}-2e^{-5t}\\e^{-5t}\end{bmatrix}$$
$$= \begin{bmatrix}\ln|t| + \frac{8t}{5} - \frac{8}{25}\\ 2\ln|t| + \frac{16t}{5} + \frac{4}{25}\end{bmatrix}$$

Solution:

$$\mathbf{x}(t) = \mathbf{x_0} + \mathbf{x_p} = A\begin{bmatrix}1\\2\end{bmatrix} + Be^{-5t}\begin{bmatrix}-2\\1\end{bmatrix} + \begin{bmatrix}\ln|t| + \frac{8t}{5} - \frac{8}{25}\\ 2\ln|t| + \frac{16t}{5} + \frac{4}{25}\end{bmatrix}$$

10.3.9

Here $f(x) = x$ and $T = \pi$, so the Fourier series for x is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

where:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)dx = 0$$

because $x\cos(nx)$ is an odd function and $(-\pi, \pi)$ is symmetric about the origin

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx \\
&= \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx \qquad \text{because } x\sin(nx) \text{ is even} \\
&= \frac{2}{\pi}\left(\left[x\left(\frac{-\cos(nx)}{n}\right)\right]_0^{\pi} - \int_0^{\pi}\frac{-\cos(nx)}{n}\right) \qquad \text{integration by parts} \\
&= \frac{2}{\pi}\left(\frac{-\pi\cos(n\pi)}{n} + 0 + \left[\frac{\sin(nx)}{n^2}\right]_0^{\pi}\right) \\
&= \frac{2}{\pi}\left(-\frac{\pi}{n}(-1)^n + 0\right) \\
&= \frac{2}{n}(-1)^{n+1}
\end{aligned}$$

Therefore the Fourier series is:

$$\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin(nx)$$